John Crossley, Esq
with the author's Complts

NEW FORMULAS

FOR THE

LOADS AND DEFLECTIONS

OF

SOLID BEAMS AND GIRDERS.

BY

WILLIAM DONALDSON, M.A., A.I.C.E.,
AUTHOR OF 'SWITCHES AND CROSSINGS,' AND A 'TREATISE ON OBLIQUE ARCHES.'

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CONTENTS.

INTRODUCTION.

The aim of the Author, when he commenced the Essay which he now ventures to submit to the criticism of his fellow-engineers, was to obtain a set of formulas for the loads and deflections of solid beams, strictly based on the assumption *ut tensio sic vis*, but not vitiated by the absurd restriction to the full meaning of the maxim, which seems hitherto to have been the stumbling-block in the way of any honest investigation of the subject, *viz.* that the neutral surface must in all cases pass through the centre of gravity of every section of the beam.

The clear and simple meaning of the maxim is, that if any given force is capable of producing a certain amount of extension or compression, double the force would produce double the amount of extension or compression; not that the absolute amounts of extension or compression produced by the same force would be equal—an assumption which is identical with the assumption that the neutral surface passes through the centre of gravity of each section.

The earlier efforts of the scientific men who undertook the investigation of this subject, were simply directed to find out formulas for the loads which beams are capable of supporting without injuring the elasticity of the materials of which they are composed, and the amount of deflection due to those loads; in other words, to find out the intensity of the internal molecular forces without troubling themselves about theories, as the way in which these molecular forces acted and reacted upon each other—a plan which the Author feels sure the practical sound common sense of all engineers speaking the English language, whether English or American, will acknowledge to

stress in either the upper or lower flange will never be caused by a working load, fully one-sixth part of the whole weight of iron used has been wasted, considerations which the Author hopes will secure for his Essay either a prompt refutation, if the principles inculcated in it are wrong, or a silent acknowledgment of their truth evinced in the altered practice of engineers.

NEW FORMULAS

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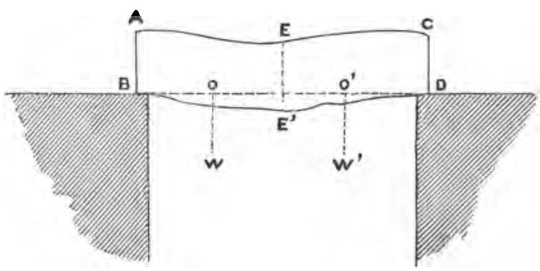
CHAPTER I.

THEORETICAL INVESTIGATION OF THE FORMULAS.

LET A B C D (Fig. 1) be a heavy beam resting on two points of support, one at each end, at the same level and loaded in any manner between those points B and D of support. Suppose this beam divided into any two parts A B E E′ and C D E′ E by a vertical plane at right angles to its length, and let W, W′

FIG. 1.



be the weights of the portions A B E′ E and C D E′ E respectively and the intercepted loads. Since a vertical plane through the centres of gravity of the two parts must pass through the points of support B and D, vertical lines through the same centres of gravity must also cut the horizontal line joining B and D.

Let the vertical lines through the centres of gravity of

A B E′ E cut the line B D in O and of C D E′ E in O′, then we shall have

$$\text{Pressure on pier D} = \frac{\text{W.O B} + \text{W}'.\text{O}'\text{B}}{\text{B D}},$$

$$\text{Pressure on pier B} = \frac{\text{W.O D} + \text{W}'.\text{O}'\text{D}}{\text{B D}}.$$

Now, if we consider the conditions of equilibrium of each part of the beam separately, it is evident that the action and reaction of one part upon the other part must be such that, taking the portion A B E′ E, the sum of the vertical components of the internal stresses, the pressure of the pier at B and the weight of the portion A B E′ E must be, having regard to sign, zero, and also the sum of the moments of the reaction of the pier B, of the weight of A B E′ E and of the resultant or resultants of the internal stresses, round any horizontal line at the section at E E′ must be zero too. The same remark applies, *mutatis mutandis*, to the part E E′ D C.

Now the form of fracture ascertained by experimentally loading a beam with its breaking weight, shows that the fibres of the upper part of the beam are in compression, and of the lower in tension, and that the fibres at the extreme top and bottom of the beam are in a state of greater compression and extension respectively than those nearer the centre.

These facts, ascertained by experiment, lead naturally to the three following assumptions:—

1st. Along some line in the plane of section, which may be either straight or curved, there is no stress parallel to the direction of the beam.

This line is called the neutral axis of the section.

2nd. The intensity of the component of the internal stress at any point in the section, resolved parallel to the direction of the beam, varies as the distance of that point from the neutral axis.

3rd. The sums of the components of the internal stresses at every point of the section, resolved perpendicularly to the direction of the beam, must be equal to the difference between the pressure on the pier and the weight of the intervening beam and load.

The intensity of this component of the internal stress may be supposed to be the same at every point in the section.

If the above assumptions be correct, the direction of the internal stress at different points in the beam must be as follows:—

1st. The direction of the internal stress at any point, in any section, lies in a vertical plane parallel to the vertical plane through the points of support.

2nd. In any section the stress at the neutral axis is either zero or acts vertically.

3rd. In the sections close to the points of support, the direction of the stress at every point is vertical.

4th. In that section which divides the beam into two parts, such that the reaction of either pier is equal to the weight of the intervening beam and its load between that pier and the section, the internal stress is at every point horizontal.

5th. In any section between these two the acute angle, which the direction of the stress at any point between the neutral axis and the top or bottom of the beam makes with the vertical, increases as the distance of the point from the neutral axis increases.

The only way to test the correctness of the three assumptions just stated with regard to the nature of the internal stresses on any section of the beam is to compare the results of experiments with the results obtained by calculation from formulas based on those assumptions, which may be done as follows:—

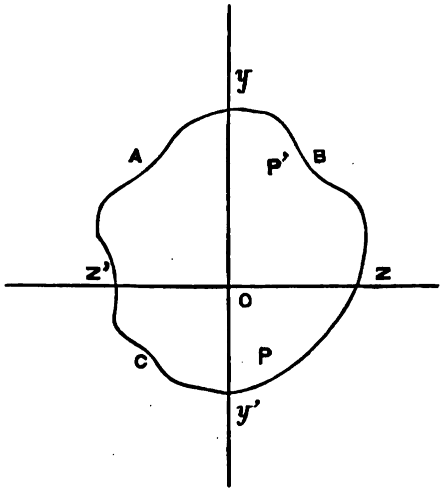
Application of the Assumptions.

Let A B C (Fig. 2) be a section of a beam resting on two orizontal supports at a distance x from one of them. To simplify the investigation we will suppose that the load is uniformly distributed, that the weight of an unit of length of the beam is the same at every point, and that the points of support are on the same level, and also that the form of the section is such that the neutral axis is a straight line.

The formulas obtained will apply to the case of a beam in an inclined position, only in this last case the bending forces will

be the components of its weight and load, resolved at right angles to its length. The components parallel to its length will be supported by the beam as a pillar.

FIG. 2.



Take $Z' O Z$ the neutral axis for one axis of co-ordinates, and $y O y$ perpendicular to it and passing through the centre of gravity of the section for the other.

Let z, y be the co-ordinates of any point P′ above the neutral axis.

Let $z, -y$, be the co-ordinates of any point P below the neutral axis.

Let c, c' be the distances of the neutral axis from the bottom and top edges respectively.

Let f, f' be the horizontal components of the stress per unit of area on the extreme bottom and top edges respectively actually exerted.

Let w be the weight per unit of length of the beam and load.

Let l be the distance between the points of support.

Then we shall have by assumption No. 2,

Horizontal component of stress at P′ in compression $= \dfrac{y f'}{c'}$,

Horizontal component of stress at P in tension .. $= \dfrac{y f}{c}$,

Pressure on either pier .. .. $= \dfrac{l w}{2}$,

Weight of portion length x .. .. $= w x$.

If we take moments round the neutral axis, we have, giving opposite moments, opposite signs.

Moment of reaction of pier round neutral axis

$$= \frac{l w x}{2},$$

Moment of weight of length x round neutral axis

$$= - \frac{w x^2}{2};$$

Moment of stresses in compression round neutral axis

$$= - \frac{f'}{c'} \int \int y^2 \, d y \, d z,$$

Moment of stresses in tension round neutral axis

$$= - \frac{f}{c} \int \int y^2 \, d y \, d z.$$

The integration extending over the areas in extension and compression respectively. Therefore for equilibrium we must have

$$\frac{l w x}{2} - \frac{w x^2}{2} - \frac{f}{c} \int \int y^2 \, d y \, d z - \frac{f'}{c'} \int \int y^2 \, d y \, d z = 0. \qquad (1)$$

Resolving the forces horizontally, we have, neglecting the friction of the pier,

$$\frac{f}{c} \int \int y \, d y \, d z = \frac{f'}{c'} \int \int y \, d y \, d z. \qquad (2)$$

The integration extending between the same limits.

In the above investigation it must be carefully borne in mind that f, f' are the extreme stresses actually exerted, not necessarily, when the breaking load is applied, the extreme stresses in the case of both f and f', which the material can bear, nor under any other load does the relation

$$\frac{f}{f'} = \frac{\text{ultimate tensile strength}}{\text{ultimate compressive strength}}$$

necessarily exist.

In equation (2) the expressions under the sign of integration are equal to the product of the areas under compression and extension, multiplied by the distances of their respective centres of gravity from the neutral axis. Consequently if A, A'

represent those areas and D, D′ the corresponding distances, we shall have

$$\frac{\mathrm{A\,D}}{\mathrm{A'\,D'}} = \frac{f'c}{fc'}; \tag{3}$$

also if d be the depth of the beam,

$$c + c' = d. \tag{4}$$

With the above equations we have to solve the following propositions :—

(1). Given the length of the beam, the form of section at every point, the values of f and f' actually exerted, to find the magnitude of the load.

(2). Given the length of the beam, the form of section at every point, and the magnitude of the load, to find the value of f and f' actually exerted.

(3). Given the length of the beam, the values of f and f' actually exerted, and the magnitude of the load, to determine what form of section will satisfy the equations.

By means of equations (1) (2) and (4) we can at once arrive at a solution of Prop. I. With regard to Prop. II., since we have only three equations and four unknown quantities f, f' c, c', we shall therefore only obtain equations of relation between any two of them.

Prop. III. cannot be solved in its general form.

Now as regards Prop. I., how have the values of f, f', assumed to be known, been ascertained? All that we know about them is, what values neither of them can exceed, since these correspond with the values of the ultimate tensile and compressive strength of the material, which have been ascertained by experiments, and also, if the maxim *ut tensio sic vis* holds true, that the ratio

$$\frac{\text{total tensile force actually exerted}}{\text{total compressive force actually exerted}}$$

bears a constant ratio to the ratio

$$\frac{\text{total increment produced in an unit of length}}{\text{total decrement produced in an unit of length}};$$

and also that the ratio of the successive increases of the forces

bears a constant ratio to the ratio of the successive increments and decrements.

In tables of the strength of materials there is usually a column headed E, the modulus of elasticity. This modulus of elasticity is the weight which is capable of extending a bar of any material of one square inch sectional area to double its length. There are of course very few materials so extensible as to permit this value to be ascertained by actual experiment, it has therefore to be obtained by calculation from the elongations produced by stresses within the limits of elasticity. In calling this the modulus of elasticity of the material, it has been tacitly assumed that a given weight would produce the same amount of elongation as compression, an hypothesis which is certainly not based on the results of experiments.

In the following investigation the modulus of elasticity in extension is still denoted by the letter E, whilst E' is used to denote the modulus of elasticity in compression.

Within whatever limits of stress the maxim *ut tensio sic vis* holds true, within the same limits the values of E and E' are constant, and the relation subsisting between them and c, c' f and f' can be easily found.

The equations just given refer only to beams of such a section that the neutral axis is a straight line. This would plainly be the case with all the conic sections, if the neutral axis passed through the centre of figure; but when the neutral axis does not bisect the depth of the beam, it can only be a straight line in the case of rectangular beams with one side vertical, and flanged girders with the transverse sections of the tops and bottoms of the beams horizontal.

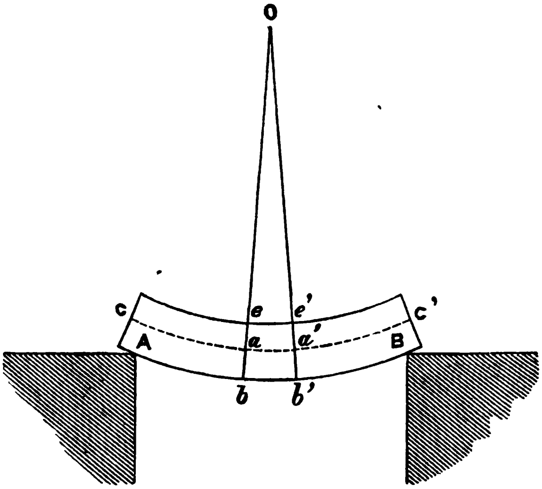
To these two cases therefore our investigations must be confined.

Let A B (Fig. 3), a rectangular beam or flanged girder, be deflected by any given weight. Let C a a' C be the line of intersection of a vertical plane, which passes through the points of support and bisects the beam, with the neutral surface; then C a a' C will be the locus of the centres of the neutral axes of every section, and may be either straight or curved.

Since the upper edge of the beam is parallel to the lower

edge, the curvature of the two edges after deflection will be similar, and a normal to either will also be a normal to the other, the difference between the radii being equal to the depth of the beam. Let O be the centre of curvature of the point b. Through the point b and a contiguous point b' draw the normals $O\,e\,a\,b$, $O\,e'\,a'\,b'$, cutting the upper edge in the point e, e' and the neutral line in the points a, a', respectively.

FIG. 3.



Now whether the neutral line $C\,a\,a'\,C$ be straight or curved, we may consider that the length $a\,a'$ is equal to the natural lengths of $b\,b'$ and $e\,e'$ before extension and compression respectively, when the interval $b\,b'$ is indefinitely diminished. If ρ be the radius of curvature of the lower edge of the beam, and the

$$\angle\, b\,O\,b' = \delta\,\theta,$$

we shall have

$$b\,b' = \rho\,\delta\,\theta,$$
$$e\,e' = (\rho - c - c')\,\delta\,\theta,$$

and $a\,a'$ will be between the values

$$(\rho - c)\,\delta\,\theta \text{ and } (\rho - c \pm \delta\,c)\,\delta\,\theta,$$

and therefore ultimately we shall have

$$a\,a' = (\rho - c)\,\delta\,\theta.$$

From these equations we deduce

$$\frac{\rho - c - c'}{\rho - c} = \frac{e\,e'}{a\,a'},$$

$$\frac{c'}{\rho - c} = \frac{a\,a' - e\,e'}{a\,a'} = \frac{f'}{E'};$$

also

$$\frac{\rho}{\rho - c} = \frac{b\,b'}{a\,a'},$$

$$\frac{c}{\rho - c} = \frac{b\,b' - a\,a'}{a\,a'} = \frac{f}{E}.$$

Therefore

$$\frac{f}{f'} = \frac{c\,E}{c'\,E'}.$$

We must now apply these results to each case separately.

Rectangular beams.

If b be the breadth of the beam, the equation of moments becomes

$$w\,(l\,x - x^2) = \frac{2\,f\,b\,c^2}{3} + \frac{2\,f'\,b\,c'^2}{3},$$

and the equation

$$\frac{A\,D}{A'\,D'} = \frac{f'\,c}{f\,c'}$$

reduces to

$$\frac{c'}{c} = \frac{f}{f'} = \frac{c\,E}{c'\,E'};$$

whence

$$\frac{c'}{c} = \frac{f}{f'} = \sqrt{\frac{E}{E'}};$$

by means of this relation and the equation

$$c + c' = d,$$

we get

$$c = \frac{d}{1 + \frac{f}{f'}} = \frac{d}{1 + \sqrt{\frac{E}{E}}},$$

$$c' = \frac{\frac{d f}{f'}}{1+\frac{f}{f'}} = \frac{d\sqrt{\frac{E}{E'}}}{1+\sqrt{\frac{E}{E'}}}.$$

Substituting these values of c and c' in equation (1) and reducing, we get either

$$w(l x - x^2) = \frac{2 f b d^2}{3\left(1+\frac{f}{f'}\right)} = \frac{2 f b d^2}{3\left(1+\sqrt{\frac{E}{E'}}\right)};$$

or

$$w(l x - x^2) = \frac{2 f' b d^2}{3\left(1+\frac{f'}{f}\right)} = \frac{2 f' b d^2}{3\left(1+\sqrt{\frac{E'}{E}}\right)};$$

the formulas in terms of f or f' to be used according as E is $\lessgtr$ E′.

Since

$$\frac{c'}{c} = \frac{f}{f'} = \sqrt{\frac{E}{E'}},$$

it follows that within whatever limits of stress the maxim *ut tensio sic vis* holds true, for values of f or f' within those limits the neutral surface retains an invariable position, and is parallel to the upper and lower surface of the beam. Consequently the line $C a a' C$ will be a straight line before the beam has been deflected, and if we take this line therefore for axis of x, and a vertical line through either extremity for axis of y, the equation to the line $C a a' C$ after the beam has suffered deflection will be of the form

$$y = f(x a),$$

and the values of y, when x is put equal to $\frac{l}{2}$, will give the central deflection.

Now $\rho - c$ is the radius of curvature of the line $C a a' C$ at the point a; if x, y be co-ordinates of this point, we shall therefore have

$$\rho - c = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}$$

$$= \frac{1}{\frac{d^2 y}{dx^2}},$$

nearly, since the deflection is always so small that we may put $\left(\frac{dy}{dx}\right)^2 = o$: therefore

$$\frac{d^2 y}{dx^2} = \frac{f}{\mathrm{E}\,c} = \frac{f'}{\mathrm{E}'\,c'}$$

$$= \frac{f\left(1 + \sqrt{\frac{\mathrm{E}}{\mathrm{E}'}}\right)}{\mathrm{E}\,d}, \text{ if } f \text{ be } \angle\, f',$$

$$= \frac{f'\left(1 + \sqrt{\frac{\mathrm{E}'}{\mathrm{E}}}\right)}{\mathrm{E}'\,d}, \text{ if } f \text{ be } 7\, f'.$$

Now f, f' are functions of x, if the breadth as well as depth remains constant, and therefore must be expressed in terms of x before the integration is performed. There are two cases.

1st. Load distributed where w is the weight per unit of length of the beam and load.

$$f = \frac{3\,w\,(l\,x - x^2)\left(1 + \sqrt{\frac{\mathrm{E}}{\mathrm{E}'}}\right)}{2\,d^2\,b},$$

$$f' = \frac{3\,w\,(l\,x - x^2)\left(1 + \sqrt{\frac{\mathrm{E}'}{\mathrm{E}}}\right)}{2\,d^2\,b}.$$

Therefore substituting and performing the integrations we get for the equation to the curve C $a\,a'$ C either

$$y = \frac{3\,w\left(1 + \sqrt{\frac{\mathrm{E}}{\mathrm{E}'}}\right)^2}{2\,\mathrm{E}\,d^3\,b}\left(\frac{l\,x^3}{6} - \frac{x^4}{12} - \frac{l^3\,x}{12}\right),$$

or expressed in terms of $f\,f'$

$$y\,(l\,x - x^2) = \frac{f\left(1+\sqrt{\frac{E}{E'}}\right)^2}{12\,E\,d}(2\,l\,x^3 - x^4 - l^3\,x)$$

$$= \frac{f'\left(1+\sqrt{\frac{E'}{E}}\right)}{12\,E'\,d}(2\,l\,x^3 - x^4 - l^3\,x).$$

Putting $x = \frac{l}{2}$ we get for the central deflection, neglecting the sign,

$$y = \frac{5\,w\left(1+\sqrt{\frac{E}{E'}}\right)^2 l^4}{128\,d^3\,b\,E}$$

$$= \frac{5\,f\left(1+\sqrt{\frac{E}{E'}}\right)l^2}{48\,E\,d}$$

$$= \frac{5\,f'\left(1+\sqrt{\frac{E'}{E}}\right)l^2}{48\,E'\,d}.$$

2nd. Central load. If W be the load applied at the centre, and w the weight per unit of length of the beam, the moment of the applied forces will be

$$\frac{W\,x}{2} + \frac{w\,(l\,x - x^2)}{2};$$

and therefore

$$f = \frac{3\left(1+\sqrt{\frac{E}{E'}}\right)}{2\,d^2\,b}\{W\,x + w\,(l\,x - x^2)\},$$

$$\frac{d^2\,y}{d\,x^2} = \frac{3\left(1+\sqrt{\frac{E}{E'}}\right)^2}{2\,E\,d^3\,b}\{W\,x + w\,(l\,x - x^2)\},$$

$$y = \frac{3\left(1+\sqrt{\frac{E}{E'}}\right)^2}{2\,E\,d^3\,b}\left\{W\left(\frac{x^3}{6} - \frac{l\,x^2}{4}\right) + w\left(\frac{l\,x^3}{6} - \frac{x^4}{12} - \frac{l^3\,x}{12}\right)\right\};$$

therefore neglecting the sign, the central deflection becomes

$$y = \frac{\left(1+\sqrt{\frac{E}{E'}}\right)^2 l^3}{128\,E\,d^3\,b}(8\,W + 5\,l\,w).$$

Since the weight $\frac{l\,w}{2}$ placed at the centre will produce the same stress as the whole distributed weight of the beam at the centre,

$$W + \frac{l\,w}{2} = \frac{4\,f\,d^2\,b}{3\,l\left(1+\sqrt{\frac{E}{E'}}\right)},$$

$$y = \frac{\left(1+\sqrt{\frac{E}{E'}}\right)f\,l^2}{12\,E\,d} + \frac{l^4\,w\left(1+\sqrt{\frac{E}{E'}}\right)^2}{128\,d^3\,b\,E};$$

similarly,

$$y = \frac{\left(1+\sqrt{\frac{E'}{E}}\right)f'\,l^2}{12\,E'\,d} + \frac{l^4\,w\left(1+\sqrt{\frac{E'}{E}}\right)^2}{128\,d^3\,b\,E'}.$$

From this we see that the deflection produced by a central load is to the deflection produced by a distributed load of equal magnitude as 8 : 5.

If the breadth of the beam be varied, so that the stresses f, f' remain constant, we have

$$\frac{d^2 y}{d x^2} = \frac{f\left(1+\sqrt{\frac{E}{E'}}\right)}{E\,d},$$

$$y = \frac{f\left(1+\sqrt{\frac{E}{E'}}\right)}{2\,E\,d}(x^2 - l\,x),$$

$$= \frac{f'\left(1+\sqrt{\frac{E'}{E}}\right)}{2\,E'\,d}(x^2 - l\,x);$$

therefore for a central section

$$y = \frac{f\,l^2\left(1+\sqrt{\frac{E}{E'}}\right)}{8\,E\,d} = \frac{f'\,l^2\left(1+\sqrt{\frac{E'}{E}}\right)}{8\,E'\,d}$$

The expressions for the deflections of rectangular beams when given in terms of the load and not of the stresses produced, are apparently of wrong dimensions, since they are of the form

$$y = \frac{C l^4}{d^3 b}$$

when the load is uniformly distributed and of the form

$$y = \frac{C l^3}{d^3 b}$$

when a central load is applied, where C is some constant. They are, however, not so in reality. The apparent discrepancy arises from the different ways in which the load and the total internal stresses are expressed. Thus the applied loads are expressed in one case in units of length of the beam, and in the other simply in the same unit as the stress f per square inch, whilst the total stress is expressed in units of area.

Thus, if the applied load had been stated in units of area of the upper surface of the beam, we should have had the total load in either case $= l\,b\,w$, where w is the weight per unit of area, and then

$$y = \frac{C l^4}{d^3},$$

which is symmetrical. From this we see that so long as C, which is a function of the breadth and the total load, remains constant, the ratio of the deflection to the length varies as the cube of the ratio of the length to the depth. When the stress f remains constant, the ratio of the deflection to the length varies simply as the ratio of the length to the depth.

Flanged Girders.

The integral in the equation of moments may be evaluated with sufficient exactness in the following manner. The case of a wrought-iron girder with top plates and angle-irons is chosen, because the solution includes that of a simple cast-iron beam with top and bottom flanges.

In both cases the moment of the web is left out of considera-

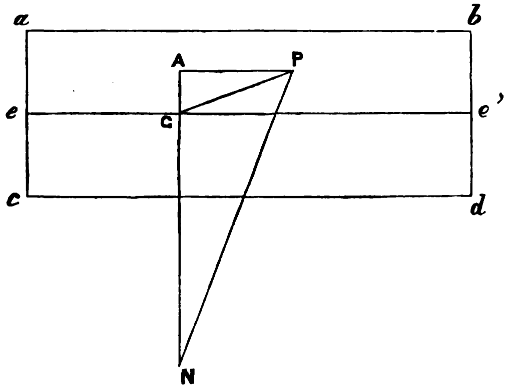
tion, because its amount is insignificant compared with those of the booms or flanges.

Let b, t be the breadth and depth of top plate, b', t' those of the horizontal part of the angle-iron and the included web, and b'', t'' of the vertical part of the angle and the included web.

The evaluation of the integral $\iint y^2\, dy\, dz$ for the whole of the three areas is equal to sum of the evaluations of each separately, which may be obtained in this way.

Let $a\,b\,d\,c$ (Fig. 4) represent any one of the three areas. Let $e\,G\,e'$ be a line through its centre of gravity G parallel

FIG. 4.



to the neutral axis, and A G N another line at right angles to these two, meeting the neutral axis in N. Let P be any point in the area, whose ordinate referred to the neutral axis is y. Join PN, P G, and draw P A perpendicular to N G meeting N G, produced in A, then we shall have, since

$$\begin{aligned} NA &= y, \\ AG &= \text{distance of P from axis } e\,G\,e' \\ &= y', \text{ suppose} \\ NA^2 &= (NG + AG)^2 = NG^2 + AG^2 + 2\,AG\,.\,GN, \\ y^2 &= NG^2 + y'^2 + 2\,NG\,.\,y'; \end{aligned}$$

or

$$\iint y^2\, dy\, dz = NG^2 \iint dy\, dz + \iint y'^2\, dy\, dz,$$

since

$$2\,NG \iint y'\, dy\, dz = o.$$

Also since

$$\iint dy\,dz = \text{area},$$

$$\iint y'^2\,dy\,dz = \text{area} \times \frac{(\text{depth})^2}{12},$$

we get the following evaluations for each of the separate areas:

Of top plate

$$= b\,t\left\{\frac{t^2}{12} + \left(c' - \frac{t}{2}\right)^2\right\};$$

Of horizontal part of top angle-iron and included web

$$= b'\,t'\left\{\frac{t'^2}{12} + \left(c' - t - \frac{t'}{2}\right)^2\right\};$$

Of vertical part of top angle-iron and included web

$$= b''\,t''\left\{\frac{t''^2}{12} + \left(c' - t - t' - \frac{t''}{2}\right)^2\right\}.$$

If therefore a, a', a'' represent the areas of the sections of the top plate, the horizontal part of the top angle-iron and the included web, and the vertical part of the top angle-iron and the included web respectively, and d, d', d'' be the distances of their respective centres of gravity from the neutral axis, we get, neglecting those terms which involve the second and higher powers of t, t', t'',

$$b\,t\left\{\frac{t^2}{12} + \left(c' - \frac{t}{2}\right)^2\right\} = b\,t\,c'\left(c' - \frac{t}{2}\right) = a\,d\,c',$$

$$b'\,t'\left\{\frac{t'^2}{12} + \left(c' - t - \frac{t'}{2}\right)^2\right\} = b'\,t'\,c'\left(c' - t - \frac{t'}{2}\right) = a'\,d'\,c',$$

$$b''t''\left\{\frac{t''^2}{12} + \left(c' - t - t' - \frac{t''}{2}\right)^2\right\} = b''\,t''\,c'\left(c' - t - t' - \frac{t''}{2}\right) = a''d''c'.$$

Therefore within these limits of exactness for the top flange

$$\begin{aligned}\iint y^2\,dy\,dz &= (a\,d + a'\,d' + a''\,d'')\,c' \\ &= \mathrm{A}'\,\mathrm{D}'_1\,c',\end{aligned}$$

if A' be the area of the top flange and D'_1 the distance of the centre of gravity of that area from the neutral axis. Similarly, if $\mathrm{A}\ \mathrm{D}_1$ be the area and the distance of the centre of gravity of

that area from the neutral axis of the bottom flange we shall have for that flange

$$\int\int y^2\, dy\, dz = A\, D_1\, c\,;$$

consequently

$$w\,(l x - x^2) = 2 f'\, A'\, D'_1 + 2 f\, A\, D_1.$$

Now the equation

$$\frac{A\, D_1}{A'\, D'_1} = \frac{f'\, c}{f\, c'}$$

becomes for flanged beams

$$\frac{A}{A'} = \frac{f'}{f} = \frac{c'\, E'}{c\, E} = \frac{D'_1\, E'}{D_1\, E},$$

since the ratio $\frac{c}{c'} = \frac{D_1}{D'_1}$ is practically one of equality, and therefore

$$f\, A = f'\, A',$$

and

$$w\,(l x - x^2) = 2 f\, A\, (D_1 + D'_1)$$
$$= 2 f A\, D = 2 f'\, A'\, D,$$

where D is the distance between the centres of gravity of the areas of the top and bottom flanges, and f, f' are the stresses actually exerted. It is evident therefore that, of girders of different sections in which the distance between the centres of gravity of the top and bottom flange areas and the sum of those areas are constant, that section will be strongest in which A, A′ are so proportioned as to call forth the maximum values of f and f'.

From the equations

$$\frac{c'}{c} = \frac{E\, A}{E'\, A'},$$

$$c + c' = d,$$

we obtain

$$c = \frac{d}{1 + \frac{E\, A}{E'\, A'}}, \qquad c' = \frac{d}{1 + \frac{E'\, A'}{E\, A}};$$

therefore

$$\frac{d^2 y}{d x^2} = \frac{f\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)}{\mathrm{E}\,d} = \frac{f'\left(1+\frac{\mathrm{E'\,A'}}{\mathrm{E\,A}}\right)}{\mathrm{E'}\,d}.$$

If A, A′ remain constant, f and f' will be functions of x, and must be expressed in terms of x before the integration is performed. There will therefore be the two cases of distributed and central load.

1st. Distributed load

$$f = \frac{w\,(l\,x - x^2)}{2\,\mathrm{A\,D}} \qquad f' = \frac{w\,(l\,x - x^2)}{2\,\mathrm{A'\,D}},$$

$$\frac{d^2 y}{d x^2} = \frac{w\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)}{2\,\mathrm{E\,A\,D}\,d}\,(l\,x - x^2),$$

$$y = \frac{w\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)}{2\,\mathrm{E\,A\,D}\,d}\left(\frac{l\,x^3}{6} - \frac{x^4}{12} - \frac{l\,x^3}{12}\right);$$

and for the central deflection, neglecting sign,

$$y = \frac{5\,w\,l^4\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)}{384\,\mathrm{E\,A\,D}\,d} = \frac{5\,w\,l^4\left(1+\frac{\mathrm{E'\,A'}}{\mathrm{E\,A}}\right)}{384\,\mathrm{E'\,A'\,D}\,d};$$

or expressed in terms of f, f' the extreme stresses at the central section

$$y = \frac{5\,l^2 f\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)}{48\,\mathrm{E}\,d} = \frac{5\,l^2 f'\left(1+\frac{\mathrm{E'\,A'}}{\mathrm{E\,A}}\right)}{48\,\mathrm{E'}\,d}.$$

2nd. Central load.

Following the method adopted in the case of a rectangular beam, we get, for the constant deflection,

$$y = \frac{l^3\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)}{384\,\mathrm{E\,A\,D}\,d}\,(8\,\mathrm{W} + 5\,l\,w)$$

$$= \frac{l^2 f\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)}{12\,\mathrm{E\,D}} + \frac{l^4\left(1+\frac{\mathrm{E\,A}}{\mathrm{E'\,A'}}\right)w}{384\,\mathrm{E\,A\,D}\,d}$$

$$= \frac{l^2 f' \left(1 + \frac{E' A'}{E A}\right)}{12 E' d} + \frac{l^4 \left(1 + \frac{E' A'}{E A}\right) w}{384 E' A' D d}.$$

If the areas A, A′ be varied so that f, f' remain constant, we have

$$y = \frac{f\left(1 + \frac{E A}{E' A'}\right)}{2 E d} (x^2 - l x)$$

$$= \frac{f' \left(1 + \frac{E' A'}{E A}\right)}{2 E' d} (x^2 - l x).$$

Therefore for a central deflection

$$y = \frac{f l^2 \left(1 + \frac{E A}{E' A'}\right)}{8 E d}$$

$$= \frac{f' l^2 \left(1 + \frac{E' A'}{E A}\right)}{8 E' d}.$$

Hence if the areas be so proportioned that

$$\frac{f}{f'} = \frac{E}{E'},$$

we shall have

$$\frac{A E}{A' E'} = \frac{f A}{f' A'} = 1;$$

and the central deflection, when f is constant, is

$$y = \frac{f l^2}{4 E d} = \frac{f' l^2}{4 E' d};$$

and when A + A′ is constant

$$y = \frac{5 f l^2}{24 E d} = \frac{5 f' l^2}{25 E' d}.$$

Mons. Gaudard, in a paper read at the Institute in Session 1868–69 on the Resistance of Materials, speaking of the shape of the transverse section before and after flexure, states, that if

the transverse section before flexure be a rectangle (Fig. 5), it would be represented by a curvilinear trapezium (Fig. 6), after

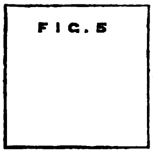


FIG. 5

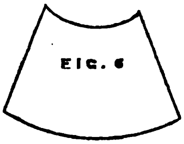


FIG. 6

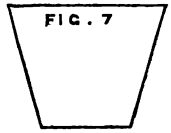


FIG. 7

flexure with a concave narrow side uppermost, because, to use his own words, "we know that an extended fibre contracts transversely, whilst a compressed fibre swells," a very good reason for supposing just the contrary, *viz.* that the rectangle becomes a trapezium, with the broad side uppermost (Fig. 7). He goes on to state, "When a prismatic solid is subjected to a bending movement, which varies from one section to another, the flexure is said to be unequal. There are then produced slidings between the sections and even between the fibres. The longitudinal slidings between the fibres develop among themselves tangential reactions or frictions of adherence energetic in the neighbourhood of the central fibre, but which decrease and vanish in the fibres of the external contour, which are supposed free. This law of the slidings compels the transverse sections, which were originally plane, to undulate in curved surfaces, cutting normally the external faces of the solid. Further, the slidings between the different threads of particles produce inverse effects of such a nature that no alteration occurs in the longitudinal stresses of the fibres, which enables us to study the flexure on the simplifying hypothesis of the sections remaining plane, when we have only in view the tensions and compressions of the fibres." With each and all of these opinions the author of this essay cannot help disagreeing. As to the sliding between the sections due to the shearing stress, a comparison of the examples in Table I., Chapter II., in which the breaking weights of various cast-iron beams are given, shows that the shearing stress per square inch in sections close to the abutments of square cast-iron beams, whose length between supports is not less than thirteen times the depth, does not exceed 800 lbs. per square inch, which taking the transverse sliding

equal to the longitudinal decrement in an unit of length due to an unequal compressive force, cannot cause an amount of sliding greater than one twelve-thousandth part of an inch. In some materials the sliding will exceed, in others be less than, this amount, but in none will the deviation from it be of such an extent as to affect the present discussion. The more remote the section is from a point of support, the less is the shearing stress, whilst at the centre there is none at all. As to the slidings between the fibres, the imaginary concave outline of the upper edge is probably attributed to a supposed vertical transverse sliding of the fibres, though why the external fibres should slide over the internal ones, if any sliding takes place at all, it is not easy to conceive, since all fibres at the same distance from the neutral axis are exposed to the same force tending to make them leave or approach the neutral axis. Now, if p', p represent this force per square inch on the extreme fibres of the top and bottom of the beam, we shall have

$$p = \frac{f}{\rho} \qquad p' = \frac{f'}{\rho};$$

or, if we suppose the moduli of elasticity equal,

$$p = p' = \frac{2f^2}{\mathrm{E}d}.$$

Now, on the supposition of the equality and constancy of the moduli for cast-iron beams, the maximum value of E is about 14,000,000 lbs., and of E′ about 13,000,000 lbs. per square inch; consequently the maximum value of this stress is equal to $\frac{30}{d}$ lbs. only. The intensity of the stress per square inch, which tends to cause one fibre to slide over another in a longitudinal direction, must, like that of the vertical sliding stress, be greatest at the extreme top and bottom fibres, and therefore the actual reactions between the fibres is of greater intensity than those between the fibres near the centre. Now the maximum intensity of this stress will be best exhibited by an example in which the whole stress is produced by a central load, since it is the difference between the longitudinal stresses at contiguous

sections which alone can produce this sliding. We will take then a bar of cast iron 16 feet long. The breaking stress per square inch at the extreme fibres will be about 14,000 lbs., and if the weight of the beam be neglected, the stress at a contiguous section one-hundredth part of an inch distant will be to this in the ratio of 95·99 to 96, or the actual stress tending to produce sliding 1·5 lb. per square inch.

Again, the stress at a section midway between the centre and the abutment would be equal to 7000 lbs. per square inch, and the stress at a contiguous section one-hundredth part of an inch distant would be to this in the ratio of 47·99 to 48, or the actual stress tending to produce sliding 1·5 lb. per square inch, the same as before. Now these stresses tending to produce longitudinal sliding of the fibres vary inversely as the length of the beam, and are much less for a distributed load than for a central load; also on the fibres immediately below the top fibres and immediately above the bottom fibres similar stresses are exerted tending in the same direction, but less in magnitude, and so on till we reach the neutral axis. It will not therefore be the whole stress of 1·5 lb. per square inch which tends to cause the top fibres to slide over the next, but the difference between 1·5 lb. and the stress per square inch on the next set of fibres, and so on. Surely no slidings of such minute extent are capable of being taken into account, even in the most exact investigation.

If, however, they do exist, although the total compressions or extensions between contiguous sections on either side of the centre might not vary, the compression and extension on an unit at the centre would be diminished, and therefore the results arrived at on the hypothesis of plane sections would not be so exact as if this supposed sliding did not exist.

Towards the end of an investigation into the effects produced by this imaginary sliding, Mons. Gaudard gives a condition, which must hold between some of his coefficients in order that the curves of the upper and lower halves of the section may meet tangentially at the neutral axis, and adds, "Certain authors omit the condition just mentioned, and make the stresses vary uniformly, but in different rate for tension and for com-

pression. Then the figure representing the pressures will make a break upon the neutral axis," that is to say, the upper half of the section will meet the lower half, not tangentially, but at an angle. So they would if, according to the axiom laid down by Mons. Gaudard, the neutral axis passed through the centre of gravity, and retained an invariable position in the figure; but in that case the rate of extension and compression must be equal, because the curvatures of the upper and lower surfaces of the beam are similar up to the eve of fracture. It will be indisputably proved in the subsequent pages, that the rates for tension and compression do vary, that the neutral axis does not necessarily in all materials ever pass through the centre of gravity, and does not maintain an invariable position, but that it is continuously changing its position with every change in the magnitude of the stress. In order to do this, the formulas arrived at in the preceding pages will be tested by the results of such experiments as the author has been able to find on beams of cast iron, wrought iron, and timber.

CHAPTER II.

APPLICATION OF FORMULAS TO CAST-IRON BEAMS.

The attention of the author was first called to the great discrepancy, which exists between the breaking weights of rectangular beams calculated from formula based on the supposed equality of the moduli of elasticity and the actual breaking weights ascertained by experiment, by Prof. Reynolds of Owen's College about four years ago, with whom he had at the time some correspondence on this subject. This discrepancy is in every case so great, that the results given by the formula, so far as breaking weights are concerned, cannot even be looked upon as rough approximations.

The following Tables have been compiled from the result of experiments performed by Eaton Hodgkinson for the Royal Commission on the application of Iron to Railway Bridges, which are given in detail in the report published in 1849.

Table I. contains particulars of the experiments performed by breaking various beams by statical loads. In the report there is a Table, which gives the results of the experiments on the ultimate tensile and compressive strengths of all the principal kinds of iron; and in the Tables, which give the results of the experiments on the transverse breaking weights, the class of iron of which the bars were cast is stated, so that although we do not know the exact ultimate tensile and compressive strength of bars taken, so to speak, from the same pot, we know the limits between which they would in all probability lie. Column 1 in Table I. gives the average ratio of the breaking tensile stress to the breaking compressive stress; Columns 2 and 3 the maximum and minimum values of the breaking tensile stress. It will be seen that two sets of values are given for Blaenavon Iron No. 2; they correspond with the values derived from two different samples of this iron, and in every case that set of values has been used, which makes the calculated and experimental weight agree most nearly. The calculated weight here referred to is that obtained from the formula

$$w = \frac{4 f b d^2}{3 l \left(1 + \frac{f}{f'}\right)} = \frac{4 f b d^2}{3 l \left(1 + \sqrt{\frac{E}{E'}}\right)},$$

when we make

$$\frac{f}{f'} = \sqrt{\frac{E}{E'}} = \frac{\text{ultimate tensile strength}}{\text{ultimate compressive strength}}.$$

In Column 4 is given the ultimate tensile strength calculated from the above formula corresponding to the breaking weights ascertained by actual experiments, which are given in Column 5. In those cases where this calculated value does not fall within the maximum and minimum values given in Columns 2 and 3, the breaking weight calculated from the above formula by making f equal to the mean between the two extreme values, is given in Column 6, whilst in Column 7 is given the breaking weights calculated from the formula

$$W = \frac{2 f b d^2}{3 l},$$

which is obtained on the supposition that the moduli of elasticity are equal, f having the value given in Column 4, when that lies between the extreme values. When it does not, f is put equal to the mean value.

In the two first experiments the beam was broken by a force applied horizontally; in the remainder by a vertical weight, so that the weights given in Column 5, with the exception of the two first, include half the weight of the bar between the supports.

TABLE I.

1.	2.	3.	4.	5.	6.	7.	Description of iron used in each experiment, form of section of bar, and interval between supports.	
$\frac{\text{Ultimate tensile strength}}{\text{Ultimate compressive strength}}$	Maximum tensile breaking stress in lbs. per square inch.	Minimum tensile breaking stress in lbs. per square inch.	Breaking tensile stress from the formula in lbs. per square inch $f = \frac{3lW\left(1+\frac{f}{f'}\right)}{4d^2b}$	Central breaking weight in lbs. ascertained by experiment.	Central, breaking weight in lbs. from the formula $W = \frac{4fd^2b}{3l\left(1+\frac{f}{f'}\right)}$	Central breaking weight in lbs. from the formula $W = \frac{2fd^2b}{3l}$		ft. in. in. in.
·152	18,488	15,459	16,968	819	819	472	Blaenavon,	No. 2, 13 6×3×1½
”	”	”	16,395	402	402	232	”	” 9 0×2×1
·209	15,836	13,299	14,596	2685	2685	1622	”	” 13 6×3×3
”	”	”	15,582	1274	1274	769	”	” 9 0×2×2
·152	18,488	15,459	20,855	447	364	210	”	” 4 6×1×1
·210	13,718	11,059	15,448	1261	..	..	Low Moor,	No. 1, 9 0×2×2
·161	16,216	14,468	14,823	1261	1261	758	”	No. 2, 9 0×2×2
·210	29,814	23,287	27,275	2230	2230	1347	Mr. Stirling's iron	No. 2, 9 0×2×2
”	”	”	30,873	630	557	337	”	” 4 6×1×1
·162	24,053	22,760	18,021	1545	..	..	”	No. 3, 9 0×2×2
”	”	”	23,561	505	505	291	”	” 4 6×1×1
·179	14,309	12,927	19,641	411	285	168	Bowling,	No. 2, 4 6×1×1
·192	15,760	13,270	17,778	372	300	179	Brymbo,	No. 1, 4 6×1×1
·200	16,776	14,326	20,558	423	320	192	”	No. 3, 4 6×1×1
·172	15,129	13,274	19,603	413	300	176	Yniscedwyn,	No. 1, 4 6×1×1

From the above Table we see that on every experiment with bars over 4 ft. 6 in. long, with two exceptions, the calculated values of f lie within the extreme limits. In one of the cases the iron has probably by mistake been classed as No. 1 Low Moor iron instead of No. 2 Low Moor iron, since the calculated value given in the succeeding row lies between the extreme values.

The other exception is that of a bar of Mr. Morries Stirling's iron No. 3, 9′ 0″ × 2″ × 2″, the experimental breaking weight of which is so small that the calculated value of f is actually 4740 lbs. less than the minimum experimental value. In all the other cases the calculated value, when not lying between the two, exceeds the maximum. If the breaking weight of this bar be compared with that of a bar of Mr. Stirling's No. 2 iron of the same size, and likewise the ultimate mean tensile strengths of the two irons, the breaking weights are to each other as 22 : 15 in round numbers, whilst the tensile strengths are as 26 : 23. The bar of No. 3 iron had probably therefore some unperceived defect.

In the case of the bars 4 ft. 6 in. long, on the contrary, with but one exception, the values of f calculated from the formula are from 3 to 30 per cent. in excess of the experimental values, and, strange to say, that exception is a bar of Mr. Morries Stirling's iron No. 3, which gave such anomalous results in the case of the bar 9′ 0″ × 2″ × 2″.

A comparison of the results of the experiments shows that in eight of the examples the breaking weights calculated from the formula, which involves the value of the ratio E : E′, agree with those derived from experiment, if we suppose that on the eve of fracture the ultimate tensile and compressive resistances of the material are called into action, and that in the remaining six the calculated weights from the same formula compared with those obtained from the ordinary formula, may be looked upon as close approximation to the actual breaking weights. Are we therefore to conclude that the principles on which the formula is based are exactly true, and the discrepancies in the case of the 4 ft. 6 in. bars are simply owing to the rules not holding exactly true on the eve of fracture?

If so, then, since $\frac{f}{f'} = \sqrt{\frac{E}{E'}}$ and $\frac{E}{E'}$ is by hypothesis constant, the ratio $\frac{f}{f'}$ must always be equal to the ratio of the ultimate tensile to the ultimate compressive strength, whatever be the absolute values of f and f'. Conversely the ratio of the moduli of elasticity must have a constant value equal to the square of

the ratio of the ultimate tensile to the ultimate compressive strength, or as 36 : 1 nearly.

Table II. contains the results of experiments on the extension and compression of bars of cast iron; the bars experimented on were 10′ 0″ × 1″ × 1″, and in the experiments on tensile strength several bars were coupled together, but the results in the Table are reduced to those of a bar 10′ 0″ × 1″ × 1″.

Columns 1 in each division give the magnitude of the stress per square inch. Columns 2, the corresponding extensions and decrements in inches; and Columns 3 the corresponding sets. Columns 4 give the uniform moduli of elasticity, that is to say, the modulus calculated on the supposition that the total actual elongation produced by any stress has been produced uniformly, following the law *ut tensio sic vis;* and Columns 5 the instantaneous moduli of elasticity, that is to say, the moduli calculated from the difference between the increments and decrements in the length produced by the differences between the stresses corresponding to various values of the absolute magnitude of the total stress applied.

TABLE II.

1.	2.	3.	4.	5.	1.	2.	3.	4.	5.
EXTENSION. Mean Values from Four Bars.					COMPRESSION. Mean Values from Four Bars.				
Weight in lbs.	Extension in inches.	Set in inches.	Uniform Modulus of Elasticity in lbs.	Instantaneous Modulus of Elasticity in lbs.	Weight in lbs.	Extension in inches.	Set in inches.	Uniform Modulus of Elasticity in lbs.	Instantaneous Modulus of Elasticity in lbs.
2,108	·0186	·0005	13,600,000	..	2,065	·0188	·0005	13,181,000	..
3,161	·0287	·0011	13,217,000	12,512,000	4,129	·0388	·0023	12,770,000	12,384,000
4,215	·0391	·0018	12,936,000	12,162,000	6,194	·0598	·0040	12,428,000	11,800,000
5,269	·0500	·0027	12,646,000	11,604,000	8,259	·0788	·0065	12,577,000	13,042,000
6,323	·0613	·0037	12,377,000	11,193,000	10,324	·0995	·0085	12,463,000	11,971,000
7,376	·0734	·0052	12,059,000	10,443,000	12,388	·1203	·0109	12,348,000	11,908,000
8,430	·0859	·0066	11,777,000	10,118,000	14,453	·1416	·0141	12,248,000	11,634,000
9,484	·0995	·0084	11,438,000	9,300,000	16,518	·1634	·0171	12,131,000	11,415,000
10,538	·1136	·0106	11,132,000	8,970,000	18,583	·1851	·0205	12,047,000	11,420,000
11,592	·1283	·0130	10,842,000	8,604,000	20,647	·2063	·0248	12,010,000	11,738,000
12,645	·1448	·0161	10,479,000	7,665,000	24,777	·2497	·0322	11,907,000	11,419,000
13,700	·1668	·0210	9,856,000	5,755,000	28,907	·2970	·0430	11,577,000	10,475,000
14,793	·1859	·0241	9,549,000	6,867,000	33,031	·3534	·0610	11,227,000	8,745,000

A comparison of the results given in the Table shows—

1st. That both the uniform and instantaneous moduli of elasticity in extension and compression decrease as the stress increases.

2nd. That the moduli of elasticity in extension decrease with the increase of stress much more rapidly than the corresponding moduli in compression.

3rd. That in the case of tensile stresses up to a stress of 12,000 lbs. per square inch, which is about six-sevenths of the ultimate breaking stress, the decrements in the moduli of elasticity vary almost exactly as the corresponding increments of stress, the decrement being about 300,000 lbs. for every increment of 1050 lbs., whilst beyond the limit of 12,000 lbs. per square inch of stress the rate of decrement in the moduli is very rapid.

4th. That in the case of compressive stresses the law of uniform decrements in the moduli corresponding to uniform increments in the stress does not hold good for stresses greater than 25,000 lbs. per square inch, which is only about one-quarter of the ultimate breaking stress, and that during the same interval the rate of variation is not very regular, being about 100,000 lbs. per square inch, corresponding to an increment of 2065 lbs. in the stress; also that the results of only two experiments for stresses greater than 25,000 lbs. per square inch are given, *viz.* for 29,000 lbs. and 33,000 lbs., the decrements corresponding to these increments being respectively 230,000 lbs. and 450,000 lbs.

From the results of the experiments given in Table II., Hodgkinson obtained the following relations between the stresses and increments and decrements on length of a bar, l, inches long, in which e is the increment and d the decrement in inches corresponding to the stresses f and f' per square inch in lbs.:—

$$f = 13{,}934{,}000\,\frac{e}{l} - 2{,}907{,}432{,}000\,\frac{e^2}{l^2},$$

$$f' = 12{,}931{,}500\,\frac{d}{l} - 522{,}979{,}200\,\frac{d^2}{l^2}.$$

The experiments were not carried further in the case of the compressive stresses for the following reason. It was discovered

on the application of the stress next greater than 33,000 lbs. that the results obtained were illusory, owing to the bar failing as a pillar. Now although the bar did not give visible indications of failure as a pillar before the application of a stress greater than 33,000 lbs. per square inch, it is probable that it really began to fail when the irregular jump occurred in the ratio of the decrement in the modulus to the increment of the stress, and that in fact the successive decrements in length throughout are greater than they would have been if the sectional area had been greater. That the relation between f' and d obtained from the results of the experiments is not correct, may be easily proved by comparing the maximum value of f' derived from the formula, which should be equal to the ultimate crushing strength, with the actual crushing strength ascertained by experiment. The value from the formula is 80,000 lbs. per square inch.

Now the eight bars with which the experiments, the mean results of which are given in Table II., were made, were cast, two of each sort, from Low Moor No. 2, Blaenavon No. 2, Gartsherrie No. 3, and a mixture of Leeswood and Gartsherrie. The crushing strength of the Low Moor iron is given in the report as 95,000 lbs. per square inch; of Blaenavon No. 2, there are two samples, the crushing strength of one sample being less, and of the other greater than that of Low Moor No. 2. In these experiments, however, the modulus of elasticity of the Blaenavon bars was greater than that of the Low Moor bars, and therefore they must have been of the best sort, the crushing strength of which is 110,000 lbs. per square inch. The crushing strength of neither of the two last irons is given, but since their moduli of elasticity are both greater than that of the Low Moor, and the modulus of one of them greater than that of the Blaenavon bars, the mean crushing strength of the four could not be less than the mean between those of Low Moor No. 2 and Blaenavon No. 2, which is 103,000 lbs. per square inch. If we determine the relations between f' and d by means of the results of the first eleven weights and the corresponding decrements, we find

$$f' = 12{,}780{,}960\,\frac{d}{l} - 464{,}774{,}400\,\frac{d^2}{l^2},$$

which only gives a breaking weight of 91,000 lbs.: we may therefore fairly conclude that the experiments made to ascertain the modulus of elasticity in compression have not given satisfactory results.

Although the maxim *ut tensio sic vis* does not exactly hold true within any limits of stress in the case of cast iron, since the difference between the least and greatest values of the moduli of elasticity does not exceed 8 per cent. of the value of the greater modulus for tensile stresses between 2000 lbs. and 8000 lbs., and for compressive stresses between 2000 lbs. and 25,000 lbs., the formulas based on the assumption of the constancy of the values of the moduli of elasticity, ought to give results within 4 per cent. of the truth for proof and working loads. The moduli of elasticity for proof and working stresses are besides so nearly equal to each other, that the formula based on the assumption of their constancy and equality will give practically correct proof and working loads, although the ultimate breaking loads obtained from the same formula cannot be looked upon as even remote approximations to the truth.

Conversely it is evident that it would be wrong to use formulas with constants determined by ascertaining the experimental breaking weights to design beams intended to support proof and working loads.

Although the breaking weights ascertained from the formula

$$W = \frac{4 f b d^2}{3 l \left(1 + \frac{f}{f'}\right)}$$

are practically correct, the relation established between f, f' and the moduli of elasticity, *viz.* $\frac{f}{f'} = \frac{\sqrt{E}}{\sqrt{E'}}$, is utterly wrong when the stresses exerted approach their ultimate limits. This admits of a very easy explanation. In accordance with the maxim *ut tensio sic vis*, the stress at any point whose distance from the neutral axis is y, is equal to $\frac{y f}{c}$ or $\frac{y f'}{c'}$ according as it is below or above the neutral axis, whereas it is in reality equal to either $\frac{a e}{l} - \frac{p e^2}{l^2}$ or $\frac{a' d}{l} - \frac{p' d^2}{l^2}$, in which a, p a', p' are con-

stants determined in the way already explained, and $l = a\,a'$ (Fig. 3). Now the investigation into the relation between the radii of curvature of the upper and lower surfaces of the beam, the corresponding stresses and increments and decrements of an unit of length of the bottom and top fibres, the depths $c\,c'$ and the corresponding moduli of elasticity does not depend upon the truth of the maxim *ut tensio sic vis*, if we are to understand by this that the ratio of the increment or decrement of length to the stress producing it, is constant for all values of the stress. Also the relations established between the increments and decrements of an unit of length and the distances of the top and bottom fibres of the beam from the neutral axis are true also for those of the intermediate fibres at any distance from the neutral axis, and we have therefore

$$\frac{e}{l} = \frac{y}{\rho} = \frac{d}{l},$$

and the equation of moments becomes

$$\frac{W(lx-x^2)}{2} = \int_0^c \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\frac{ay^2}{\rho} - \frac{py^3}{\rho^2}\right) dy\,dz + \int_0^{c'} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\frac{a'y^2}{\rho} - \frac{p'y^3}{\rho^2}\right) dy\,dz,$$

where $\frac{1}{\rho} = \frac{f}{Ec} = \frac{f'}{E'c'}$, and $c + c' = d$.

Since the values of E and E′ corresponding to the stresses f and f' are supposed to have been ascertained by actual experiment, by means of the above relations we should obtain a formula, which might be considered theoretically correct, in terms of the stresses of the extreme fibres and corresponding moduli.

Now, although the moduli of elasticity for stresses near the centre are greater than for those near the edges, the extensions towards the latter exceed those towards the centre in a much greater degree. Therefore the stresses exerted so much exceed the stresses towards the centre, that the moment of the stress at each point will still vary very nearly as the square of the distance of that point from the neutral axis, and therefore the sum of the moments of the stresses towards the edges constitutes the greater part of the whole moment of the section. Conse-

quently formulas in terms of the stresses at the extreme sections, founded on the assumption *ut tensio sic vis,* may be expected to give, as experiment shows that they do give, practically correct results.

Now the relation between the stress f and f' and the moduli of elasticity was obtained by combining the relations

$$\frac{1}{\rho} = \frac{f}{Ec} = \frac{f'}{E'c'}$$

with the equation of horizontal forces. In this last equation the energy exerted by the internal stresses does not vary, as in the equation of moments very nearly as the square of the distance from the neutral axis, but only approximately as the distance simply; therefore the omission of the variation in the value of the moduli, when the stresses vary from zero to the ultimate breaking value, would introduce a serious error.

The discrepancy between the calculated and breaking values of the bars 4′ 6″ × 1″ × 1″ and the experimental values may be explained in the same way as follows:—The absolute extensions of the extreme fibres are the same whatever may be the depth of the beam; consequently the extensions of the fibres at the same distance from the neutral axis would vary inversely as the depth of the beam, and therefore those stresses near the centre whose moduli of elasticity are greatest, would be called into more active operation in shallow than in deep beams, and so cause greater deviation between the observed and calculated results.

The correct equation for the horizontal forces is

$$\int_0^c \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\frac{a\,y}{\rho} - \frac{p\,y^2}{\rho^2}\right) d\,y\,d\,z = \int_0^{c'} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\frac{a'\,y}{\rho} - \frac{p'\,y^2}{\rho}\right) d\,y\,d\,z;$$

if we integrate this and combine it with equations

$$\frac{1}{\rho} = \frac{f}{Ec} = \frac{f'}{E'c'}$$

we shall get the following relation between f, f', E and E′,

$$\frac{f^2}{E^2}\left(\frac{a}{2} - \frac{pf}{3E}\right) = \frac{f'^2}{E'^2}\left(\frac{a'}{2} - \frac{p'f'}{3E'}\right).$$

For stresses under the proof stress we may look upon the formula as practically correct, even as regards the relation $\frac{f}{f'} \doteq \sqrt{\frac{E}{E'}}$. A reference to Table II. shows that we may consider the moduli in extension and compression as of uniform value and practically equal to each other for tensile stresses under 7000 lbs. and compressive stresses under 25,000 lbs., and therefore the working and proof loads determined from the formula $W = \frac{2fbd^2}{3l}$ would be practically equal to those determined from the formula $W = \frac{4fbd^2}{3l\left(1+\sqrt{\frac{E}{E'}}\right)}$, but only equal to about three-fifths of those calculated from formulas with coefficients determined from experimental breaking weights, since in the last case the calculated proof and working loads would be to the breaking load in the ratio of the proof and working stresses respectively to the breaking stress.

In the equation of relation between the radius of curvature and the stress and corresponding uniform modulus of elasticity, *viz.*:—

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} = \frac{f\left(1+\sqrt{\frac{E}{E'}}\right)}{Ed},$$

both E and E′ as well as f are functions of x, when the value of f is not constant, and therefore must be expressed in terms of x before the integration is performed. The resulting expression would be too complicated for integration. When therefore the stresses and corresponding moduli vary so much that we cannot look upon E and the corresponding ratio $\sqrt{\frac{E}{E'}}$ as constant, it is impossible to ascertain theoretically the deflection of rectangular beams of constant section.

For values of f below the proof stress we have shown that practically we may look upon these values as constant, and

take $E = E'$. In that case the equation for the deflection of the beam becomes

$$\frac{d^2 y}{d x^2} = \frac{2f}{E d};$$

therefore the central deflection for a distributed load, if we put $l W = W$, is

$$y = \frac{5 l^3 W}{32 E b d^3},$$

$$y = \frac{5 f l^2}{24 E d}.$$

For a central load, if W be equal to the applied load, and W′ the weight of the beam,

$$y = \frac{l^3 (8 W + 5 W')}{32 E b d^3},$$

$$y = \frac{l^2 f}{6 E d} + \frac{l^3 W'}{32 E b d^3}.$$

In the case of flanged girders the error introduced by adopting the maxim *ut tensio sic vis* is practically of no consequence either in the equation of moments or of horizontal stresses, because the only stresses of which any account is taken are the extreme stresses near the top and bottom edges of the beam.

In fact the formulas for the load in terms of the stress produced, *viz.* :—

For a distributed load,

$$W = 8 f A D = 8 f' A' D,$$

or for a central load,

$$W = 4 f A D = 4 f' A' D,$$

do not involve the relative values of the moduli: since therefore $f A = f' A'$, that section of beam in which the sum of the areas $A + A'$ is constant will give the greatest working proof or breaking load in which the ratio $\frac{A}{A'}$ is equal to the ratio of the working proof or breaking tensile stress to the working proof or

breaking compressive stress respectively. Professor Rankine, after stating that the limit of proof stress ought to be determined by that stress, which first produces an increased set on reapplication, gives a Table of ratios of ultimate to proof and working stress, and of proof to working stress for various materials of construction, in which these ratios are stated to be the same for both tensile and compressive stresses, and therefore for all loads we ought to have, in accordance with that Table,*

$$\frac{A}{A'} = \frac{\text{ultimate compressive strength}}{\text{ultimate tensile strength}},$$

which is the rule deduced by Fairbairn from his experiments on the transverse breaking strength of various sections of flanged cast-iron girders.

Professor Rankine, however, does not give the data upon which his Table is based, and if we compare his statement with the results of Hodgkinson's experiments, we shall find ourselves on the horns of a dilemma. Either the experiments are not to be relied on, or his Table is not. If we refer to Table II., we find that for all stresses up to 8000 lbs. per square inch, the sets produced are the same for the same stresses both in extension and compression; we may therefore infer that the first set in either case would be caused by the same stress. Consequently on the hypothesis once entertained that materials of construction ought not to be subjected to a stress which produced a permanent set, the ratio $A : A'$ ought to be one of equality. We find, however, from the same Table, that after the limit of 8000 lbs. per square inch is exceeded, that the ratio of the set to the stress rapidly increases in the case of tensile stresses, whilst in the case of compressive stresses it continues pretty uniform up to a stress of 25,000 lbs. per square inch, and then begins rapidly to increase. At these identical stresses likewise in each case the ratio of the decrement in the uniform modulus to the increment of the stress begins rapidly to increase, and we may conclude therefore that the proof stress in neither case exceeds, but is probably less, than these limits, which we may fairly assume, however, bear to their respective proof stresses the same ratio. *If therefore the results in Table II. are to be relied on, the*

ratio A : A′ *ought not to exceed* 25,000 : 8000 *or* 3 : 1 *nearly, whilst the average value of the ratio of the ultimate breaking stresses is about* 6 : 1.

Whatever may be the values of the intensities of the stresses and whatever may be the relative values of A and A′ for the reasons already stated, the relations obtained between the radius of curvature f, f', E and E′ on the hypothesis *ut tensio sic vis*, are in the case of flanged girders practically correct. We have therefore for all intensities of stress,

$$\frac{d^2 y}{d x^2} = \frac{1}{\rho} = \frac{f\left(1 + \frac{A\,E}{A'\,E'}\right)}{E\,d}$$

$$= \frac{f'\left(1 + \frac{A'\,E'}{A\,E}\right)}{E'\,d}.$$

Now E and E′ are functions of f and f', which are connected by the relation $f\,A = f'\,A'$; we may therefore express the right-hand member of the equation in terms of f only or f' only, and if the resulting function of f or f', when these are variable and expressed in terms of x, were integrable, we should obtain a formula for the deflection which would be practically correct for all stresses. The function so obtained is, however, too complicated for integration; we cannot therefore when the sum of the areas $A + A'$ does not vary so as to make f, f' constant obtain a formula for the deflection in the case of flanged girders, unless either the ratio of the areas, which varies inversely as that of the stresses, is so chosen that the ratio $\frac{E}{E'}$ is constant, or unless the moduli are equal to each other.

If we refer again to Table No. II., we find that for stresses of small intensity the moduli of elasticity are nearly equal, but gradually diverge as the stresses increase, and also that moduli of elasticity in extension for stresses from 7000 lbs. to 9000 lbs. per square inch are equal respectively to the corresponding moduli in compression for stresses from 9000 lbs. to 28,000 lbs. If therefore we make the ratio A : A′ = 3 : 1 about, it is evident

that the ratio $E : E'$ will be very nearly one of equality within these limits. Now E, E' are functions of f and f' of the form

$$af - pf^2 \text{ and } a'f' - p'f'^2,$$

where a, p, a', p', are constants, which must be determined from the relations between f, f', E and E′ given in Table II. We have therefore

$$\frac{d^2 y}{dx^2} = \frac{1 + \frac{A}{A'}}{d(a - pf)} = \frac{1 + \frac{A'}{A}}{d(a' - p'f')},$$

which become for a distributed load

$$\frac{d^2 y}{dx^2} = \frac{2A(A + A')D}{A'd\{2ADa - pa(lx - x^2)\}}$$
$$= \frac{2A'(A + A')D}{Ad\{2A'D'a' - p'w(lx - x^2)\}},$$

and for a central load

$$\frac{d^2 y}{dx^2} = \frac{2A(A + A')D}{A'd\{2ADa - pWx\}}$$
$$= \frac{2A'(A + A')D}{Ad\{2A'Da' - p'Wx\}}.$$

Both these functions of x are very easily evaluated, but the resulting formula would be somewhat cumbrous, and therefore for practical purposes formulas in which E, E′ are supposed constant and equal to their mean value are sufficiently exact, since the mean value is within 5 per cent. of the maximum and minimum values. On the assumption that E and E′ are constant for all stresses within the proof stress in each case we get

$$\frac{d^2 y}{dx^2} = \frac{f(A + A')}{A'Ed} = \frac{f'(A + A')}{AE'd},$$

and for the central deflection putting $Dd = d^2$, due

1st, to a distributed load,

$$y = \frac{5Wl^3(A + A')}{384EAA'd^2},$$

$$y = \frac{5l^2f(A + A')}{48A'ED} = \frac{5l^2f'(A + A')}{48AE'D};$$

2nd, on a central load, W = applied load and W′ = weight of beam,

$$y = \frac{l^3(A+A')}{384\,E\,A\,A'\,d^2}\,(8\,W + 5\,W')$$
$$= \frac{l^2 f(A+A')}{12\,A'\,E\,d} + \frac{l^3(A+A')\,W'}{384\,E\,A\,A'\,d^2}.$$

If the sum of the areas A + A′ vary, whilst their ratio remains constant, in such a way that the extreme stresses remain constant, the following formulas will give the deflection due to any stress however great, since not only f, f' but the ratios $\frac{f}{E}, \frac{f'}{E'}$, and $\frac{E}{E'}$ remain constant also:—

$$y = \frac{f\,l^2\left(1+\frac{E\,A}{E'\,A'}\right)}{8\,E\,d} = \frac{f'\,l^2\left(1+\frac{E'\,A'}{E\,A}\right)}{8\,E'\,d}.$$

By means of these last equations we may indirectly determine the value of the modulus of elasticity in compression corresponding to different values of the stress up to the ultimate breaking limit. If the deflections caused by central loads determined from the formula

$$W = 4\,f\,A\,D - \frac{W'}{2}$$

corresponding to successive values of f be ascertained by experiment, since the values of E corresponding to all values of f have been ascertained by direct experiment, we shall obtain the values of E′ corresponding to successive values of f' between the maximum and minimum values, which are to those of f in the ratio A : A′. If therefore this ratio be inversely as the ultimate breaking stresses, we shall obtain the values of E′ corresponding to successive values of f' up to the ultimate breaking stress.

CHAPTER III.

APPLICATION OF FORMULAS TO WROUGHT-IRON BEAMS.

THE report of the Royal Commission does not contain the results of experiments on the transverse breaking weights of wrought-iron rectangular bars, although numerous experiments were made to ascertain the deflection of such bars due to loads not much exceeding the proof loads. Since the value of $1 + \frac{f}{f'}$ when the ultimate tensile and compressive stresses are exerted is about $2\frac{1}{4}$, the transverse breaking loads calculated from the two formulas cannot differ from each other more than 6 per cent., and, judging by the results of the experiments on cast-iron and timber rectangular beams, not from the actual experimental breaking weights, the breaking load ascertained from the formula based on the supposition of equality of the moduli being greater or less than the real breaking load according as the formula is expressed in terms of the tensile or compressive stress.

The question of the proof and working loads of wrought-iron rectangular beams cannot be discussed, until we have investigated the relative values of the tensile and compressive proof stresses of wrought iron and of the corresponding moduli of elasticity. The following Table gives the results of experiments on the extensions and decrements of wrought-iron bars 10′ 0″ × 1″ × 1″ of the quality denominated best, corresponding with Table II. in the case of cast iron, except that the sets due to the compressive stresses are not given.

The experiments made to ascertain the decrements in length due to compressive stresses were not continued beyond the stress of 32,600 lbs. per square inch, because, to use Hodgkinson's words, it was found that wrought iron sunk to any degree when subjected to a stress greater than this.

If we analyze the following Table we find in the case of the extensions, leaving the first result out of consideration, that not only the successive uniform moduli of elasticity, but also the

TABLE III.

1.	2.	3.	4.	5.	1.	2.	3.	4.	5.
EXTENSION.					COMPRESSION.				
Weight in lbs.	Extension in inches.	Set in inches.	Uniform Modulus of Elasticity in lbs.	Instantaneous Modulus of Elasticity in lbs.	Weight in lbs.	Extension in inches.	Set in ins.	Uniform Modulus of Elasticity in lbs.	Instantaneous Modulus of Elasticity in lbs.
2,667	·0097	..	32,901,000	..	4,855	·028	..	20,807,000	..
5,335	·0223	..	28,718,000	25,441,000	9,122	·052	..	21,085,000	21,335,000
8,003	·0341	·0003	28,163,000	27,132,000	13,158	·073	..	21,628,000	23,063,000
10,670	·0456	·0004	28,079,000	27,442,000	15,521	·085	..	21,920,000	23,630,000
13,338	·0571	·0005	28,031,000	27,800,000	17,655	·096	..	22,048,000	23,281,000
16,005	·0685	·0006	28,037,000	28,079,000	19,788	·107	..	22,191,000	23,270,000
18,673	·0799	·0008	28,044,000	28,079,000	21,916	·119	..	22,000,000	21,280,000
21,341	·0919	·0015	27,866,000	27,597,000	24,055	·130	..	22,205,000	23,335,000
24,008	·1049	·0040	27,463,000	24,628,000	26,188	·142	..	22,130,000	21,330,000
26,676	·1216	·0100	26,325,000	19,051,000	28,322	·154	..	22,070,000	21,341,000
29,343	·1546	·0314	22,711,000	9,785,000	30,455	·174	..	21,003,000	12,798,000
32,011	·2674	..	14,366,000	2,838,000	32,588	·214	..	18,226,000	6,399,000
5 min.	·2827	·1357	13,588,000	2,498,000					
34,678	·5148	·3686	8,083,000	1,294,000					
37,346	1·0995	1·0170	4,074,000	546,000					
repeated	1·1949	1·0297	3,750,000	471,000					
40,013	1·220	1·0930	3,920,000	2,656,000					
10 hrs.	1·443	..	3,320,000	1,932,000					
42,681	2·148	1·983	2,384,000	345,000					
46 hrs.	2·428	2·237	2,109,000	265,000					
45,348	2·580	2·377	2,109,000	732,000					
19 hrs.	2·606	2·403	2,088,000	689,000					
48,016	2·975	2·733	1,937,000	810,000					
11 hrs.	3·029	..	1,890,000	713,000					
50,684	4·195	31·941	1,450,000	262,000					
12 hrs.	4·227	..	1,438,000	256,000					

successive instantaneous moduli are equal for all intensities of stress up to nearly 19,000 lbs. per square inch, but that after this stress has been exceeded both moduli rapidly decrease, the decrease in the instantaneous modulus corresponding to a change of stress from 18,673 lbs. to 21,341 lbs., being nearly 400,000 lbs., and that corresponding to a change from 21,341 lbs. to 24,008 lbs., nearly 3,000,000 lbs. Again the increase in the set corresponding to the increment in the stress is uniform up to a stress of 16,005 lbs. per square inch, being equal to about one ten-thousandth part of an inch for each successive increase of 2700 lbs. For an equal increase from 16,005 lbs. to 18,673 lbs.,

it is double this amount, and not less than seven times this amount for the next equal increase from 18,673 lbs. to 21,341 lbs. From this we may infer that wrought iron retains its tensile elasticity practically unimpaired up to a stress somewhat less than 18,000 lbs. per square inch, and that the modulus of elasticity in extension for proof and working stresses is constant and equal to 28,000,000 lbs. per square inch.

In the case of the compressive stresses, the uniform moduli of elasticity vary from a minimum of 20,807,000 lbs. per square inch, corresponding to a stress of 5000 lbs., to a maximum of 22,205,000 lbs., corresponding to a stress of 24,000 lbs. per square inch; whilst the modulus corresponding to the maximum stress of 28,000 lbs. is 22,070,000 lbs., which is equal to the modulus corresponding to a stress of 17,000 lbs. per square inch. We may therefore conclude that the uniform modulus of elasticity in compression gradually increases until the stress reaches 17,000 lbs. per square inch, and that between the limits of 17,000 lbs. and 28,000 lbs. it is constant and equal to 22,000,000 lbs. per square inch, which we may take for the value of the modulus of elasticity in compression for proof and working loads. For increments of stress beyond 28,000 lbs., the corresponding decrements in the uniform modulus become at once excessive, being equal to 1,000,000 lbs. for an increase of stress of 2133 lbs., from 28,322 lbs. to 30,455 lbs., and to 2,800,000 lbs. for a like increment from 30,455 lbs. to 32,588 lbs. The values of the instantaneous moduli of elasticity for stresses under 28,000 lbs., are somewhat irregular, varying from a minimum of 21,280,000 lbs. to a maximum of 23,630,000 lbs. The minimum value corresponds to a stress of 21,916 lbs. per square inch, and the maximum to a stress of 15,521 lbs., whilst the instantaneous modulus of elasticity corresponding to the maximum stress of 28,000 lbs. is 21,340,000 lbs. Beyond this the decrements are very rapid, being 8,500,000 lbs. and 6,400,000 lbs. respectively, corresponding to two successive equal decrements of 2133 lbs., and beyond this stress so excessive that they could not be ascertained. This inequality in the values of the instantaneous modulus in compression is probably in a great measure due to the fact that the values of the decrements are

only given to three places of decimals, whilst those of the increments are given to four—a deviation from exactness which would affect the value of the instantaneous much more than that of the uniform modulus. We may therefore fairly conclude that the compressive elasticity of wrought iron is not practically impaired, if the compressive stress does not exceed 28,000 lbs., certainly not if it does not exceed 26,000 lbs. per square inch.

If then the limits of stress within which the elasticity of wrought iron is practically unimpaired are in the case of tensile stresses 19,000 lbs. and of compressive stresses 27,000 lbs. per square inch, ought not the flanges of wrought-iron girders to be so proportioned that the tensile and compressive stresses actually exerted are to each other in the ratio of 18 : 27, and not in that of the ultimate breaking stresses, which appear from the results of the experiments described in the Commissioners' reports to have no relation whatever to the values of the tensile and compressive stresses within the limits of elasticity? *If this be so, the ratio of the area of the bottom flange to that of the top ought not to be as* 4 : 5 *but as* 3 : 2.

Since the values of the moduli of elasticity are constant for proof and working stresses, the formulas for the proof and working loads and deflections of wrought-iron rectangular beams and flanged girders, obtained on the assumption *ut tensio sic vis*, are theoretically correct, since the value of $\frac{f}{f'} = \sqrt{\frac{E}{E'}}$ is equal to $\sqrt{\frac{28}{22}}$ or $1 \cdot 12$, the proof and working loads determined from the correct formula will be those determined on the assumption of the equality and constancy of the moduli as 50 : 53 nearly in the case of solid rectangles, and the deflections as 45 : 40, if the greatest value be assumed for the joint value, or as 35 : 40 if the less value be assumed for the joint value.

It has been already pointed out that when the moduli of elasticity are constant, the neutral axis retains an invariable position whatever may be the intensity of the stresses, the vertical distance of any point in it from the edges of the beam being determined by the relation

$$\frac{c'}{c} = \sqrt{\frac{E}{E'}},$$

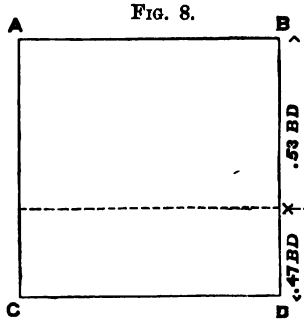
in the case of solid rectangular beams. Since any section of whatever form may be supposed to be made up of a series of parallel rectangles indefinitely narrow with the longer side vertical, it is plain that the same relation will hold for all forms of section, $c + c'$, being equal to the vertical depth of the section through the corresponding point in the neutral axis.

In the case of wrought iron $\sqrt{\frac{E}{E'}} = 1 \cdot 12$, therefore $c = \cdot 47 \times$ depth, $c' = \cdot 53 \times$ depth, and the neutral axis in the case of a rectangle with one side vertical, with a diagonal vertical, of a circle and of an ellipse will occupy the following positions shown dotted in the figures.

1st. Rectangle with one side vertical.

The neutral axis will be a straight line (Fig. 8).

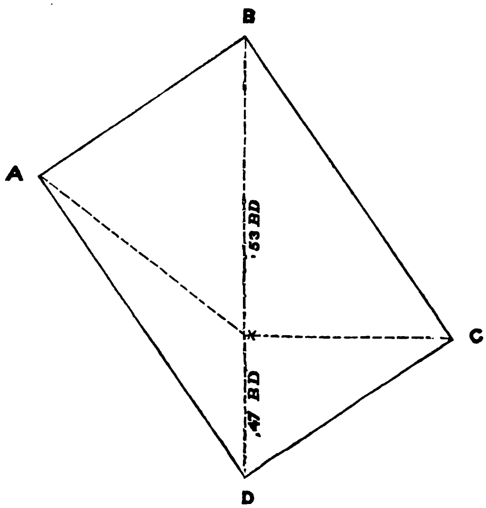
Fig. 8.



2nd. Rectangle with one diagonal vertical.

The neutral axis will consist of two straight lines (Fig. 9).

Fig. 9.



3rd. Circle radius r.

The neutral axis will be a semi-ellipse (Fig. 10), whose equation is

$$36\,x^2 + 1000\,y^2 = 36\,r^2.$$

FIG. 10.

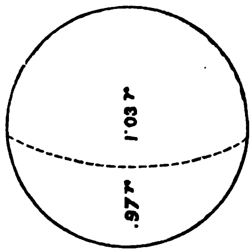
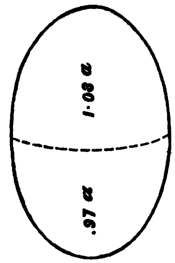


FIG. 11.



4th. Ellipse semi-axes a and b, a being vertical.

The neutral axis will be a semi-ellipse (Fig. 11), whose equation is

$$36\,a^2 x^2 + 1000\,b^2 y^2 = 36\,a^2 b^2.$$

In the case therefore of a rectangular beam with one diagonal vertical, and of beams of circular and elliptic sections, we cannot directly obtain the formulas for the proof and working loads, but we may do so indirectly by means of the artifice already alluded to, *viz.* by supposing each section divided into a number of indefinitely narrow rectangles with the long side vertical in the following manner.

The load which each rectangular slice will support is given by the formula for rectangular beams in terms of the stresses at the extreme top and bottom edges, and the weight which the whole beam will support is manifestly equal to the sum of the weights which the several parts support.

Now since the maxim *ut tensio sic vis* holds true when the moduli are constant, the stresses exerted at the extreme edges of any rectangular section are to the corresponding stresses of the central section in the ratio of the depth of the rectangle to the central depth; if therefore $2\,y$ be the depth of the rectangular section, and $\delta\,z$ its width, we shall have, d being the depth at the centre,

$$\text{Stress in compression} = \frac{2\,y\,f'}{d}$$

$$\text{Stress in extension} = \frac{2\,y\,f}{d},$$

and the distributed load supported by the rectangular slice will be

$$\frac{8\,f\,y^3\,\delta z}{3\left(1+\sqrt{\frac{\mathrm{E}}{\mathrm{E}'}}\right)d} = \frac{1 \cdot 26\,f\,y^3\,\delta z}{d};$$

and therefore the whole load supported by the beam will be equal to

$$\frac{1 \cdot 26\,f}{d} \cdot \int y^3\,d\,z.$$

Now the integral $\int y^3\,d\,z$ is identical with the integral $\int\int y^2\,d\,y\,d\,z$, when y and z are the co-ordinates of any point in the section referred to axes through the centre of gravity, the axis of y being vertical; consequently for the three last sections of beams the working and proof loads derived from the correct formula are likewise to those derived from the formula based on the equality of the moduli as 50 : 53.

Similarly, since the deflection produced by the whole of the load on the whole beam will be equal to the deflection produced on the central rectangular slice by its share of the load which caused the stress f, the formulas for the deflection in the three last cases will be of the same form as those obtained on the supposition of the equality of the moduli of elasticity, but the values of the deflections in the two cases will be as 45 : 40 or 35 : 40, according as the greater or less value of the modulus is taken for the common value.

CHAPTER IV.

APPLICATION OF FORMULAS TO TIMBER BEAMS.

TABULAR statements of the results of experiments on the transverse breaking strengths of beams made by all the recognized authorities are given in the recent issue of Tredgold, edited by Hurst, but the description of the class of wood is too vague in most cases to identify it in the Tables of cohesive and crushing strength, and the limiting values of these for different varieties of the same timber are so great, that the author has not been able to compile a Table similar to Table I. in the case of cast iron with the same degree of exactness. In the following Table no examples, except in the cases of oak and larch, are included, in which the maximum and minimum limits of ultimate stress are widely different. Since the ultimate compressive stress is less than the ultimate tensile stress, it is the value of the former which must be used in the formula, which is therefore

$$W = \frac{4 f' b d^2}{3 l \left(1 + \frac{f'}{f}\right)}.$$

In some cases, only one value is given in the Tables for the ultimate stresses. When two are given, either in one or both cases, the ratio in the first column is that of the mean values, except in the case of larch and oak. In the case of larch, only one value for the tensile stress is given, *viz.* 10,220 lbs. per square inch, and the values of the ratio in Column 1 are those of the maximum mean and minimum values of the ultimate compressive stress to 10,220 lbs. respectively. In the case of English oak, the ultimate tensile stresses vary from 9000 lbs. to 15,000 lbs. per square inch, and the ratio given is that of the minimum and maximum compressive stress to the minimum and maximum tensile stress, respectively, which are very nearly equal.

There are some very anomalous results, especially in the case of oak and larch, where the ultimate strengths range within

such wide limits. In the first example, 482 lbs. is probably a misprint for 382 lbs. There is, at any rate, no other example in which the transverse breaking weight of a beam 2′ 0″ × 1″ × 1″ exceeds 400 lbs. Notwithstanding, however, the incompleteness of the experiments to illustrate the point we are discussing, the results detailed in the following Table show that the formula for the ultimate breaking strength of rectangular beams, which involves the ratios of the ultimate breaking stresses, gives results practically correct.

TABLE IV.

1.	2.	3.	4.	5.	6.	7.	
$\frac{\text{Ultimate compressive strength}}{\text{Ultimate tensile strength}}$	Maximum compressive breaking stress in lbs. per square inch.	Minimum compressive breaking stress in lbs. per square inch.	Breaking compressive stress from formula $f = \frac{3lW\left(1+\frac{f'}{f}\right)}{4bd^2}$	Central breaking weight ascertained by experiment in lbs.	Central breaking weight in lbs. from the formula $W = \frac{4f'bd^2}{3l\left(1+\frac{f'}{f}\right)}$	Central breaking weight in lbs. from the formula $W = \frac{2f'd^2b}{3l}$	Description of timber used in each experiment, form of section of beam, and interval between supports.
							ft. in. in. in.
·67	10,058	6484	14,482	482	..	..	English oak, 2 0×1×1 young
..	..	..	11,591	382	275	228	Same experiment
..	..	..	9,926	264	264	212	English oak, 2 6×1×1 old ship
..	..	..	6,540	218	218	205	„ „ 2 0×1×1 old tree
..	..	..	10,678	282	217	194	„ „ 2 6×1×1 middling
..	..	..	823	219	219	206	„ „ 2 6×1×1 green
·43	6,782	4231	7,366	3863	2677	1915	Canadian oak, 4 0×3×3
·49	7,731	6484	8,840	4450	3577	2665	Dantzic oak, 4 0×3×3
·49	6,960	6831	7,102	212	206	153	Alder, 2 6×1×1
·55	6,819	6419	6,487	186	186	140	Spruce fir, 2 6×1×1
·65	7,518	5395	7,170	3259	3259	2672	Red pine, 4 0×3×3
·68	6,790	..	7,479	3339	3031	2546	Pitch pine, 4 0×3×3
·74	5,445	5375	6,600	2845	2331	2029	Yellow pine, 4 0×3×3
·56	6,586	5748	7,583	216	184	148	Red deal 2 6×1×1
·54	5,568	..	8,773	253	160	124	Larch (best) 2 6×1×1
·44	5,568	3201	7,206	223	138	97	„ middling 2 6×1×1
·32	..	3201	3,825	129	108	71	„ young 2 6×1×1
·83	7,277	6063	8,020	195	162	148	Walnut 2 6×1×1
·80	7,082	..	8,667	214	174	157	Sycamore 2 6×1×1

The only direct experiments on the extension and compression of timber in the direction of the fibres of which the author has been able to find any account, are those made by Mr. Kirkaldy to ascertain the decrements produced by successive increments

of strain on two beams of Riga and Dantzic fir, and those made by Mons. Chevandier and Wertheim to ascertain the moduli of elasticity in extension of various sorts of timber grown in the Vosges. The results are both given in Hurst's edition of Tredgold. In the latter case it is not explicitly asserted that the moduli were obtained by observing the extensions produced by direct tensile stresses, but they must have been so, since it is stated that the elastic limit was assumed to have been attained when the permanent set was one two-thousandth part of the length. Barlow, in his work on the strength of materials, does not discuss the question of the modulus of elasticity, although he gives the values of a constant, which he denotes by the symbol E for different kinds of timber determined by the formula

$$E = \frac{l^3 W}{32 b d^3 \delta},$$

obtained on the supposition that the maxim *ut tensio sic vis* holds true, and the tacit assumption that the moduli of elasticity are equal to each other and of constant value, by observing the deflection δ produced by successive loads. Since in the case of timber the cohesive force is always greater than the compressive force, we may conclude that the modulus in extension is greater than the modulus in compression, and therefore the value of E given by Barlow would be an approximation to one-eighth part of the value of the modulus of elasticity in compression.

In Rankine's 'Applied Mechanics,' and in Tredgold, the values of the moduli of elasticity are given for nearly every sort of timber, but not a hint is given as to how they were obtained, except in the case of the timber grown in the Vosges. Since the moduli of elasticity for Riga and Dantzic fir given in the Table are widely different from those determined from the direct experiments of Kirkaldy, as will shortly be shown, it is probable that several of them have been obtained from the values of E ascertained by Barlow. At any rate in the case of teak, ash, beech, pitch pine, Mar Forest fir, and a few others, the values of the moduli of elasticity given in the Tables are approximately equal to eight times the corresponding values of E.

In many others the tabular values are widely different from these products, and in all cases greater; they may be therefore perhaps the moduli of elasticity in extension determined by direct experiment, whilst the former are the moduli in compression.

No detail is given of the experiments of Mons. Chevandier and Wertheim, merely a tabular statement of the mean results. Those of Kirkaldy are given at length, and are abstracted in the following Table up to the point at which the beam was observed to be deflected.

The beams were each 20 feet long, and of the following dimensions :—

	One End.	Middle.	Other End.
Riga fir ..	13′5″ × 13″	13″ × 13″	13″ × 12′8″
Dantzic fir	13′5″ × 13″	13′5″ × 13′2″	13′5″ × 12′5″

Column 1 gives the actual weight applied; 2, the decrement produced; and 3, the modulus of elasticity per square inch in lbs., obtained by taking a mean value of the three areas.

TABLE V.

1.	2.	3.	1.	2.	3.
White Riga.			Red Dantzic.		
Total Pressure in lbs.	Decrements in Length in inches.	Modulus of Elasticity of Compression in lbs. per square inch.	Total Pressure in lbs.	Decrements in Lengths in inches.	Modulus of Elasticity of Compression in lbs. per square inch.
20,000	·032	880,800	20,000	·035	788,000
40,000	·083	679,000	40,000	·088	627,000
60,000	·122	693,000	60,000	·133	622,300
80,000	·158	713,000	80,000	·170	649,000

The value of the modulus of elasticity for Baltic timbers varies, according to Tredgold,

From 1,687,500 lbs. to 1,957,750 lbs. per square inch.

and, according to Rankine,

From 1,200,000 lbs. to 1,900,000 lbs. per square inch.

Between these limits also, lie the values of the moduli of elas-

ticity in extension of fir grown in the Vosges, determined by Chevandier and Wertheim from direct experiment.

The value of Barlow's constant E for Riga fir varies according to him from 166,100 to 123,800. The less value is about one-seventh of the modulus of elasticity in compression determined from Kirkaldy's experiments.

The value assigned to the same constant for red pine is 230,000, and for Memel 209,000, so that the moduli of elasticity in compression for these two are apparently nearly equal to each other, and about one and one-half times that of Riga fir, although the ultimate tensile and compressive strengths of the three are nearly equal, a discrepancy for which the author cannot find any explanation. The experiments performed by Fincham, the details of which are given in Tredgold's work, show that the ratio of the load to the deflection diminishes as the weight increases; the value of the constant E therefore is less, when determined from loads approaching the breaking loads, than for loads less than the proof load. Now the condition, which properly-constructed beams of timber are designed to fulfil, is that when subjected to a proof load the central deflection shall not exceed a given fraction of the whole length, usually a $\frac{1}{480}$th part. The formula, from which the scantlings must be determined, would therefore be, using Barlow's constant,

$$b\,d^3 = \frac{m\,l^3\,W}{32\,E},$$

obtained by putting the deflection $\delta = \frac{l}{m}$. Since the value of this constant, given in the Tables, has been calculated from the deflections produced by loads varying from one-fourth to four-fifths of the breaking load, the value of the product $b\,d^3$ is apparently in excess of the value required for a proof load. The value of E, however, has been calculated from the incorrect formula based on the assumption of the equality of the moduli of elasticity, and would therefore be to its correct values corresponding to the observed deflection approximately in the ratio of $\left(1 + \sqrt{\frac{E'}{E}}\right)^2 : 4$. One error will therefore more or less compensate for the other.

CHAPTER V.

PROPORTIONS OF WROUGHT-IRON GIRDERS.

ALTHOUGH the proper proportions of girders have no direct relevancy to the subject of this essay, they are the ultimate objects aimed at in discussions of this kind, and therefore the following short investigation of the relative proportion of length to depth to ensure a given deflection under a given load cannot be looked upon as wholly out of place. The general rule followed in designing girders is to make the depth of plate girders one-thirteenth of the span, and of lattice girders one-tenth. In the previous investigations the case of plate girders only has been discussed, since lattice girders are not solid beams, nor will the formulas given for ascertaining the loads and deflections from the known areas of the flanges the span and depth, or *vice versâ,* apply with equal accuracy to the case of lattice girders, since the stresses on the top and bottom flanges do not vary gradually from point to point in each boom, but are constant throughout each bay and have a sudden and finite difference at the point of attachment of the diagonals. The nearer these points of attachment are to each other, the more nearly will the formulas give correct results. The formulas will therefore be practically correct in the case of what we may call trellis-work girders, and may be looked upon as applicable in practice to all kinds of trussed girders with parallel horizontal booms so far as the deflection is concerned.

Since the deflection varies inversely as the depth, the difference in the ratio of the span to the depth, cannot have been adopted with a view to give the same deflection in each case for the same span, but solely with a view to economize material, the increased weight of the web in the case of plate girders being supposed to more than counterbalance the saving in the booms, when the depth exceeds one-thirteenth of the span, whilst in the case of lattice girders this does not take place until the depth exceeds one-tenth.

Now the oscillations in the roadway arising from the deflection and subsequent resilience of the girders caused by the passing

live load, are the only ones which interfere with the safety of the road and the stability of the structure, the deflection caused by the dead load being permanent and provided for by giving sufficient initial cambre to the girder. Since the ratio of the live load to the dead is not constant, but decreases as the span increases, it is plainly not right to adopt a constant ratio of span to depth, which would entail a greater proportionate deflection for short than for long spans, and consequently cause more violent oscillations both to the structure and the passing loads, but on the contrary a constant ratio of span to deflection caused by the live load.

The formula for the total deflection in terms of the stress produced is

$$y = \frac{c f l^2}{E d}.$$

Now the stresses f, f' actually produced vary directly as the weights producing them; if therefore W^l W^d be the live and dead loads, we shall have

$$\text{Deflection caused by live load} = \frac{W^l y}{W^l + W^d},$$

$$\text{Deflection caused by dead load} = \frac{W^d y}{W^l + W^d}.$$

In the following Table the ratios of span to depth are determined, so that the deflection caused by the live load may be equal to one two-thousandth part of the span.

The values of the ratios, $\dfrac{\text{live load}}{\text{sum of live and dead loads}}$, have been calculated from Mr. Baker's curves of weights of girders, the rolling load being estimated at 30 cwt. and the weight of a single line roadway exclusive of weight of main girders at 10 cwt. per foot run, so that the whole weight on a single-line girder would be 20 cwt., and on a double-line girder 40 cwt. per foot run, exclusive in each case of the weight of the girder itself.

The value of the constant C is equal to $\dfrac{5\left(1 + \dfrac{E A}{E' A'}\right)}{48}$ when

the sum of the areas $A + A'$ is constant, and to $\frac{\left(1 + \frac{EA}{E'A'}\right)}{8}$ when the sum of the areas varies, so that f and f' remain constant. The value of the greatest of these is not more than two per cent. greater than that of the least, we may therefore assume that the deflection in actual girders is equal to the mean between these two, since the sum of the areas complies with neither condition, except in the case of very short girders.

Table No. VI. refers to the case in which the ratio $\frac{A}{A'}$ has the value ordinarily assigned to it, *viz.* 4 : 5, which is practically equal to $E' : E$ or 22 : 28, so that $1 + \frac{AE}{A'E'} = 2$; and Table No. VII. to the case in which $\frac{A}{A'} = \frac{3}{2}$, so that $1 + \frac{AE}{A'E'} = 2{\cdot}91$.

In the first case, which corresponds with that of the girders of which Mr. Baker has given the weights, the value of the proof tensile stress is taken at 9 tons per square inch by that gentleman, whilst in the second it must be taken as equal to 8 tons only.

The ratio of the depth of the girders in Table No. VI. would therefore be to those in Table No. VII. in the ratio 18 : 24 nearly, if the difference of their weights be left out of consideration. Since, however, the sums of the areas of the flanges are not the same, the weights of the girders will likewise be different. Thus, if A, B be the areas of the bottom flanges in the two cases on the supposition that the depths are equal, we shall have sum of the areas in one case equal to $A + \frac{5A}{4}$ and in the other to $B + \frac{2B}{3}$, whilst A and B are connected by the relation $\frac{A}{B} = \frac{40}{45}$, so that the sum of the areas $B + \frac{2B}{3}$ will be one-sixth less than that of the areas $A + \frac{5A}{4}$. Therefore the weights of the

girders in Table No. VII. would be one-sixth less than those of the corresponding ones in Table No. VI. if the depths were the same.

Now, although the weight of the web plates and stiffeners in the case of the girders in Table No. VII. would be increased by the extra depth, the decrease in the sectional area of the flanges owing to the increase in the depth, would probably more than compensate for the increase, so that we may safely take the weights of the girders in Table No. VII. as fully one-sixth less than those of the girders in Table No. VI. The ratio of the span to the deflection caused by the live load and of the span to the initial cambre for girders in which the ratio of the span to the depth is not the same as that given in the Tables may be easily computed, since the first ratio will be to 2000, and the second to the corresponding ratio given in the Tables inversely as the given ratio of the span to the depth is to the corresponding ratio given in the Tables.

Thus, if R, C be the ratios of the span to the depth and the initial cambre respectively given in the Tables, and r the ratio of the span to the depth, in any special case, we have,

$$\frac{\text{span}}{\text{deflection due to live load}} = \frac{2000\,\mathrm{R}}{r},$$

$$\frac{\text{span}}{\text{initial cambre}} = \frac{\mathrm{C\,R}}{r}.$$

The distance of the neutral axis from the top of the girder is determined by the relation

$$\frac{c'}{c} = \frac{\mathrm{A\,E}}{\mathrm{A'\,E'}};$$

consequently the neutral axis of the girders, the flange areas of which are so proportioned that $\frac{\mathrm{A\,E}}{\mathrm{A'\,E'}} = 1$, will be exactly midway between the top and bottom of the beam; when the ratio $\mathrm{A} : \mathrm{A'} = 3 : 2$, the distance $\mathrm{C'}$ of the neutral axis from the top of the beam will be equal to $\cdot 66\,d$, or nearly seven-tenths of the depth.

In addition to the initial cambre due to the dead load, there will be that due to imperfect workmanship and the permanent cambre which must, for æsthetical reasons, be given to every girder to prevent the appearance of sagging in the centre—on neither of these points can the author give any information.

TABLE VI.

$$\frac{A}{A'} = \frac{E'}{E} = \frac{22}{28}.$$

$f = 4 \cdot 5$ tons per square inch. $f' = 3 \cdot 5$ tons per square inch.

If the deflection due to the live and dead loads be equal to $\frac{1}{1000}$th part of the span,

$$\frac{\text{Span}}{\text{Depth}} = 6 \cdot 07.$$

Distance of neutral axis from top of girder = $\cdot 5\, d$.

Span in feet.	Single-line Girder.			Double-line Girder.		
	Live load / Whole load	Span / Depth	Span to / Initial cambre	Live load / Whole load	Span / Depth	Span to / Initial cambre
20	·71	8·5	4900	·73	8·3	5408
40	·69	8·8	4450	·70	8·6	4666
60	·66	9·2	3882	·68	8·8	4250
80	·63	9·6	3404	·66	9·0	3882
100	·60	10·1	3000	·64	9·4	3554
140	·55	11·0	2444	·59	10·2	2878
180	·51	11·9	2080	·55	11·0	2444
220	·46	13·2	1702	·51	11·9	2080
260	·42	14·3	1448	·47	12·9	1774
300	·39	15·5	1280	·44	13·8	1570
350	·35	17·3	1076	·41	14·8	1390
400	·32	19·0	912	·37	16·4	1174

Table VII.

TABLE VII.

$$\frac{A}{A'} = \frac{3}{2}.$$

$f = 4$ tons per square inch. $f' = 6$ tons per square inch.

If the deflection due to the sum of the proof live and dead loads be equal to $\frac{1}{1000}$th part of the span,

$$\frac{\text{Span}}{\text{Depth}} = 4\cdot66.$$

Distance of neutral axis below top of girder $= \cdot 66\,d$.

Span in feet.	Single-line Girder.			Double-line Girder.		
	$\frac{\text{Live load}}{\text{Whole load}}$	$\frac{\text{Span}}{\text{Depth}}$	$\frac{\text{Span}}{\text{Initial cambre}}$	$\frac{\text{Live load}}{\text{Whole load}}$	$\frac{\text{Span}}{\text{Depth}}$	$\frac{\text{Span}}{\text{Initial cambre}}$
20	·72	6·5	5142	·73	6·4	5408
40	·70	6·7	4666	·71	6·6	4896
60	·67	7·0	4060	·69	6·8	4452
80	·65	7·2	3714	·67	7·0	4060
100	·62	7·5	3262	·65	7·2	3714
140	·58	8·0	2762	·61	7·6	3128
180	·53	8·7	2256	·57	8·2	2650
220	·49	9·5	1920	·54	8·7	2348
260	·45	10·3	1636	·51	9·2	2082
300	·42	11·1	1448	·48	9·7	1846
350	·38	12·2	1226	·44	10·6	1572
400	·34	13·7	1062	·41	11·4	1390

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